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Color-critical graphs and hypergraphs with few edges and no short cycles

H.L. Abbott[a], D.R. Hare[b,*], B. Zhou[c]

[a] *Department of Mathematics, University of Alberta, Edmonton, Alta., Canada T6G 2G1*
[b] *Department of Mathematics and Statistics, Okanagan University College, 3333 College Way, Kelowna, BC, Canada V1V 1V7*
[c] *Department of Mathematics, Trent University, Peterborough, Ont., Canada K9J 7B8*

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Abstract

We give constructions of color-critical graphs and hypergraphs with no cycles of length 5 or shorter and with relatively few edges.

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1. Introduction

A *hypergraph* is an ordered pair $(V, \mathscr{G})$ where V is a finite nonempty set whose elements are called *vertices* and $\mathscr{G}$ is a collection of nonempty subsets of V, whose members are called *edges*. We shall usually suppose that $V = \bigcup \mathscr{G}$ so that when we speak of the hypergraph $\mathscr{G}$ it is understood that we mean $(\bigcup \mathscr{G}, \mathscr{G})$. If for some n, $|E| = n$ for all $E \in \mathscr{G}$, then $\mathscr{G}$ is called an n-graph. A 2-graph is an ordinary simple graph. The *order* of a hypergraph is the number of its vertices and the *size* is the number of its edges. A hypergraph $\mathscr{H}$ is a subgraph of $\mathscr{G}$ if $\mathscr{H} \subseteq \mathscr{G}$.

An *r-coloring* of a hypergraph is an assignment of one of $r \geqslant 2$ colors to each vertex of $\mathscr{G}$ so that no edge has all of its vertices assigned the same color (i.e. no edge is *monochromatic*). $\mathscr{G}$ is *r-colorable* if $\mathscr{G}$ has an r-coloring and is *r-chromatic* if r is the least integer for which $\mathscr{G}$ is r-colorable. If $r \geqslant 3$, then $\mathscr{G}$ is called *r-critical* if it is r-chromatic but $\mathscr{G} - E$ is $(r-1)$-colorable for every edge E of $\mathscr{G}$. A *color-critical* hypergraph is one that is r-critical for some r.

A hypergraph $\mathscr{G}$ is called *linear* if $|E \cap F| \leqslant 1$ for all E, $F \in \mathscr{G}$, $E \neq F$. For $\ell \geqslant 2$, a *cycle of length* ℓ in $\mathscr{G}$ is a sequence $v_1, F_1, v_2, F_2, \ldots, F_{\ell-1}, v_\ell, F_\ell$, where $v_1, v_2, \ldots, v_\ell$ are distinct vertices, $F_1, F_2, \ldots, F_\ell$ are distinct edges, $v_i \in F_i \cap F_{i+1}$ for $i = 1, 2, \ldots, \ell - 1$ and

* Corresponding author. E-mail: dhare@okanagan.bc.ca.

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$v_1 \in F_1 \cap F_\ell$. Note that every 2-graph is necessarily linear and that a linear hypergraph is one in which there are no cycles of length 2.

Let n, r, ℓ be positive integers such that $n \geqslant 2$, $r \geqslant 3$, and $\ell \geqslant 2$. By an (n,r,ℓ)-graph we shall mean an r-critical n-graph having no cycles of length at most ℓ (i.e. girth at least $\ell+1$). Let $S(n,r,\ell)$ denote the set of integers m for which there exists an (n,r,ℓ)-graph of order m. It will be convenient to allow $\ell=1$ and to consider an $(n,r,1)$-graph as simply an r-critical n-graph. $S(2,3,\ell)$ is the set of odd integers $m \geqslant \ell$. This follows from the result of König stating that the only 3-critical 2-graphs are the cycles of odd length. Henceforth, it is understood that when $n=2$, the condition $r \geqslant 4$ is tacitly assumed.

It has been shown that there exists a least integer $M(n,r,\ell)$ such that if $m \geqslant M(n,r,\ell)$, then there exists an (n,r,ℓ)-graph of order m. This is proved in [1] and [11] for the case $\ell=1$, in [4] for the case $\ell=2$, and in [3] for the case $\ell \geqslant 3$.

For $m \in S(n,r,\ell)$, define

$$E(m,n,r,\ell) = \min\{|\mathscr{G}| : \mathscr{G} \text{ is an } (n,r,\ell)\text{-graph of order } m\}.$$

As remarked in [3],

$$\alpha(n,r,\ell) = \lim_{m\to\infty} \frac{E(m,n,r,\ell)}{m}$$

exists, is finite and nonzero. Thus, for fixed n, r, ℓ, $E(m,n,r,\ell)$ grows in an essentially linear fashion with m. However, the precise nature of this growth is not well understood. In the case $n=2$, no value of $\alpha(2,r,\ell)$ is known.

In [3], we showed that

$$\alpha(2,r+1,3) \leqslant \alpha(2,r,3)+1,$$

and we asked whether for $\ell \geqslant 4$,

$$\alpha(2,r+1,\ell) \leqslant \alpha(2,r,\ell)+1$$

holds. In this paper, we show that this inequality does hold for $\ell=4$ or 5.

That $\alpha(n,3,1)=1$ comes out of results of Seymour [10], Woodall [12], Liu [9] and Burstein [6]. In fact, Burstein proved that for each $n \geqslant 3$ there is an integer $m_0 = m_0(n)$ such that if $m \geqslant m_0$ then $E(m,n,3,1)=m$. In [2] it was shown that $\alpha(n,3,2)=1$ and in [3] we showed that $\alpha(n,3,3)=1$. We have also been able to show (unpublished) that $\alpha(3,3,4)=\alpha(3,3,5)=1$. However, no other values of $\alpha(n,r,\ell)$ have been determined. We showed in [2] and [3] that

$$\alpha(n,r+1,\ell) \leqslant \alpha(n,r,\ell)+1$$

holds for $\ell=1$ and 2. The second main result of this paper is that this inequality holds for $\ell=3$, 4 and 5. Unfortunately, our methods do not extend to the case $\ell \geqslant 6$. For further background information and references to the literature, the reader should see [2,3].

Two general constructions are needed before we continue.

1.1. The Hajós construction

Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be disjoint r-critical graphs ($n=2$). Let $\{a,b\}$ and $\{c,d\}$ be edges of $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively. Identify a with c, delete $\{a,b\}$ and $\{c,d\}$, and add the edge $\{b,d\}$. The resulting graph is r-critical and was described in [8]. Note that if $\mathscr{G}_1$ and $\mathscr{G}_2$ do not have cycles of length at most ℓ, then neither does the constructed graph.

1.2. The long-edge graph

The long-edge graph construction is described in [2]. We include it here for convenience. Let $k\geqslant 2$ be an integer. For $i=1,2,\dots,k$, let $m_i\in S(n,r,\ell)$ and let $\mathscr{G}_i$ be an (n,r,ℓ)-graph with m_i vertices. We suppose that the vertex sets are pairwise disjoint. Let E_i be an edge of $\mathscr{G}_i$, and let $v_i\in E_i$. Let v be a new vertex. The long-edge E is defined to be

$$E=\left(\bigcup_{i=1}^{k} E_i\backslash\{v_i\}\right)\cup\{v\}.$$

For each $F\in\mathscr{G}_i$, let

$$F'=\begin{cases}(F-\{v_i\})\cup\{v\} & \text{if } v_i\in F,\\ F & \text{if } v_i\notin F.\end{cases}$$

The long-edge graph $\mathscr{G}$ is defined as the hypergraph whose edges are

(1) the long-edge E,

(2) the edges F', $F\in\mathscr{G}_i$ for some i, $1\leqslant i\leqslant k$, $F\neq E_i$.

The long-edge graph $\mathscr{G}$ is simply the hypergraph obtained by identifying each v_i with v, and E is just the union of the E_i with v_i replaced with v. Moreover, the long-edge graph has the following easily verified properties.

(1) The long edge has size $k(n-1)+1$.

(2) $\mathscr{G}$ has order $m_1+m_2+\cdots+m_k-k+1$.

(3) $\mathscr{G}-E$ is an n-graph and by the choice of $\mathscr{G}_i$, $\mathscr{G}-E$ has girth at least $\ell+1$.

(4) $\mathscr{G}-E$ is $(r-1)$-colorable and in any $(r-1)$-coloring of $\mathscr{G}-E$, E is monochromatic.

2. Graphs

Theorem 1. $\alpha(2,r+1,\ell)\leqslant\alpha(2,r,\ell)+1$ *for* $\ell=4$ *or* 5, $r\geqslant 4$.

Proof. Let $\mathscr{H}$ be a $(2,r,\ell)$-graph with m vertices and $E(m,2,r,\ell)$ edges. Let $\mathscr{H}_1$ be obtained by applying the Hajós construction to two copies of $\mathscr{H}$. For $s\geqslant 2$, let $\mathscr{H}_s$ be the graph obtained by applying the Hajós construction to $\mathscr{H}_{s-1}$ and $\mathscr{H}$. $\mathscr{H}_s$ is a $(2,r,\ell)$-graph with $(s+1)(m-1)+1$ vertices and $(s+1)(E(m,2,r,\ell)-1)+1$ edges. If Δ is the maximum degree of $\mathscr{H}$ we may arrange that the maximum degree Δ_1 of $\mathscr{H}_s$ satisfies $\Delta_1\leqslant 2(\Delta-1)$.

Let $\mathscr{H}_s^*$ be the graph whose vertex set is that of $\mathscr{H}_s$ and in which two vertices are joined by an edge if their distance is at most $\ell-2$ in $\mathscr{H}_s$. The maximum degree of $\mathscr{H}_s^*$

is at most $\Delta_1(\Delta_1-1)^{\ell-3}$. The chromatic number k of $\mathscr{H}_s^*$ satisfies $k \leqslant \Delta_1(\Delta_1-1)^{\ell-3}+1$. Note that this bound on k does not depend on s. Let the color classes of a k-coloring of $\mathscr{H}_s^*$ be $V_1, V_2, \ldots, V_k$.

Let $\mathscr{G}$ be a $(2, r+1, \ell)$-graph with a vertices and b edges. Let $\mathscr{L}$ be a long-edge graph constructed from $k-1$ copies of $\mathscr{G}$. The long-edge E of $\mathscr{G}$ then has k vertices; denote them by $v_1, v_2, \ldots, v_k$.

Let $\mathscr{F}$ be the graph whose vertex set is $V(\mathscr{L}) \cup V(\mathscr{H}_s)$ and whose edge set consists of

(1) the edges of $\mathscr{L} - E$,

(2) the edges of $\mathscr{H}_s$,

(3) the edges of the type $v_i x$, $x \in V_i$, $i = 1, 2, \ldots, k$.

It is easy to see that $\mathscr{F}$ is $(r+1)$-chromatic. $\mathscr{F}$ may not be $(r+1)$-critical. However, every edge of $\mathscr{H}_s$ is critical. To see this, let xy be an edge of $\mathscr{H}_s$. There is an $(r-1)$-coloring of $\mathscr{H}_s - xy$ in colors $1, 2, \ldots, r-1$ and there is an r-coloring of $\mathscr{L}-E$ in colors $1, 2, \ldots, r$ in which all vertices of E are assigned color r. This gives an r-coloring of $\mathscr{H}_s - xy$. Thus, every $(r+1)$-critical subgraph of $\mathscr{F}$ contains $\mathscr{H}_s$ as a subgraph.

It is also the case that when $\ell = 3$, 4 or 5, $\mathscr{F}$ has girth at least $\ell + 1$. There are no cycles of length at most ℓ involving only edges of $\mathscr{L} - E$ or only edges of $\mathscr{H}_s$. Also, a cycle of length at most ℓ must involve at least two of $v_1, v_2, \ldots, v_k$. The worst one could have is a cycle of length 6 of the type $v_i x y v_j z w v_i$ where x and w are distinct vertices in V_i and y and z are distinct vertices of V_j.

Let $\widehat{\mathscr{F}}$ be the largest $(r+1)$-critical subgraph of $\mathscr{F}$. Then

$$|V(\widehat{\mathscr{F}})| \geqslant |V(\mathscr{H}_s)| = (s+1)(m-1)+1$$

and

$$|\widehat{\mathscr{F}}| \leqslant |\mathscr{F}| = (k-1)(|\mathscr{G}|-1) + (s+1)(E(m,2,r,\ell)-1) + 1 + (s+1)(m-1) + 1.$$

Thus,

$$\frac{|\widehat{\mathscr{F}}|}{|V(\widehat{\mathscr{F}})|} \leqslant \frac{(k-1)(|\mathscr{G}|-1)+1}{(s+1)(m-1)+1} + \frac{(s+1)(E(m,2,r,\ell)-1)}{(s+1)(m-1)+1} + 1.$$

As $s \to \infty$, the first term goes to 0 since the numerator is independent of s.

Therefore,

$$\begin{aligned} \alpha(2, r+1, \ell) &\leqslant \lim_{s\to\infty} \left(\frac{(s+1)(E(m,2,r,\ell)-1)}{(s+1)(m-1)+1} + 1 \right) \\ &= \frac{E(m,2,r,\ell)-1}{m-1} + 1. \end{aligned}$$

Since m is arbitrary, $\alpha(2, r+1, \ell) \leqslant \alpha(2, r, \ell) + 1$. $\square$

In [3] we gave a table of upper bounds for $\alpha(2, r, \ell)$ for small values of r and ℓ. These bounds were obtained from results given in [3] and the earlier literature. The

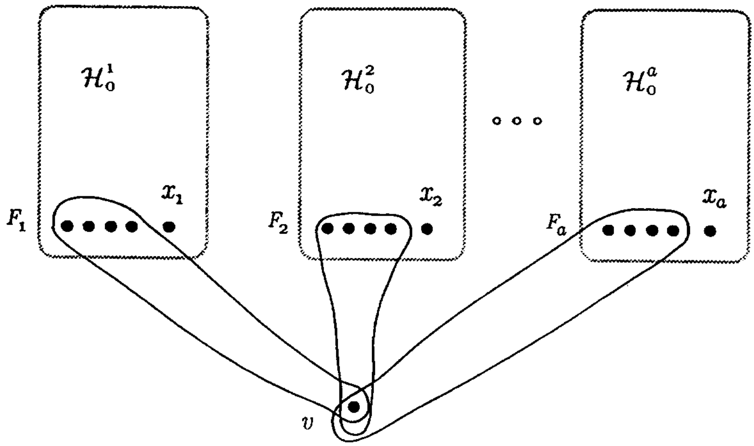

Fig. 1. The graph $\mathscr{H}_1$ (without the edges of $\mathscr{A}$).

entries beyond the first in the second and third columns of this table may now be improved by appealing to Theorem 1.

3. Hypergraphs

Theorem 1 can be extended to hypergraphs. The argument, although it is along the same lines as that used in Theorem 1, involves some additional ideas; in particular, the theorem of Erdős and Hajnal cited below.

Theorem 2. $\alpha(n,r+1,\ell)\leqslant\alpha(n,r,\ell)+1$ *for* $n\geqslant 3,\ r\geqslant 3,\ \ell=3,4,5.$

Proof. Let $\mathscr{H}_0$ be an (n,r,ℓ)-graph with m vertices and $E(m,n,r,\ell)$ edges. Let $\mathscr{A}$ be an $(n,r-1,\ell)$-graph with a vertices and b edges. If $r=3$, we have $a=n$ and $b=1$; that is, $\mathscr{A}$ consists of a single edge. Let $\mathscr{H}_0^1,\mathscr{H}_0^2,\ldots,\mathscr{H}_0^a$ be a vertex disjoint copies of $\mathscr{H}_0$. Let F_i be an edge of $\mathscr{H}_0^i$ and x_i be a vertex of F_i. Delete F_i from $\mathscr{H}_0^i$ and add $\{v\}\cup(F_i\backslash\{x_i\})$, $i=1,2,\ldots,a$. Here v is a new vertex. Form a copy of $\mathscr{A}$ on $x_1,x_2,\ldots,x_a$. Denote the resulting graph by $\mathscr{H}_1$ (see Fig. 1).

$\mathscr{H}_1$ has girth at least $\ell+1$ and the usual sort of argument shows that it is r-critical. Thus, $\mathscr{H}_1$ is an (n,r,ℓ)-graph. It has $am+1$ vertices and $aE(m,n,r,\ell)+b$ edges.

Now repeat this process. For $s\geqslant 2$, $\mathscr{H}_s$ is derived from $\mathscr{H}_{s-1}$ in the same manner that $\mathscr{H}_1$ is derived from $\mathscr{H}_0$. $\mathscr{H}_s$ is an (n,r,ℓ)-graph with $a^sm+a^{s-1}+a^{s-2}+\cdots+a+1$ vertices and $a^sE(m,n,r,\ell)+b(a^{s-1}+a^{s-2}+\cdots+a+1)$ edges.

The construction of $\mathscr{H}_s$ can be done in such a way that the maximum degree Δ of $\mathscr{H}_s$ is bounded independently of s. In fact, the x_i's can be chosen so that $\Delta\leqslant\max\{a,\Delta(\mathscr{A})+\Delta(\mathscr{H}_0)-1\}$.

Let $\mathscr{H}_s^*$ be the 2-graph whose vertex set is that of $\mathscr{H}_s$ and in which two vertices are joined by an edge if their distance is at most $\ell-2$ in $\mathscr{H}_s$. For example, if $n=4$

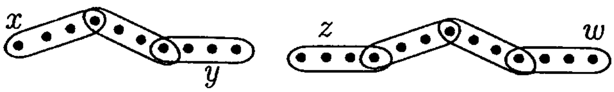

Fig. 2. Vertices x and y are joined in $\mathscr{H}_s^*$, z and w may not be.

and $\ell=5$ and x, y, z, w are as in Fig. 2, then x and y are joined by an edge in $\mathscr{H}_s^*$, but z and w may not be.

The maximum degree of $\mathscr{H}_s^*$ is at most $(n-1)\Delta((n-1)\Delta-1)^{\ell-3}$ and the chromatic number k of $\mathscr{H}_s^*$ satisfies $k\leqslant(n-1)\Delta((n-1)\Delta-1)^{\ell-3}+1$. Note that this bound on k does not depend on s. Let $V_1, V_2, \dots, V_k$ be the color classes of a k-coloring of $\mathscr{H}_s^*$.

Let $\mathscr{G}$ be an $(n, r+1, \ell)$-graph and let t be an integer to be specified later. Let $\mathscr{L}$ be the long-edge graph constructed from t copies of $\mathscr{G}$ so that the long-edge E of $\mathscr{L}$ has size $t(n-1)+1$. Choose k large pairwise disjoint subsets $W_1, W_2, \dots, W_k$ of E. We explain in a moment how 'large' is to be interpreted.

For $i=1,2,\dots,k$, let $\mathscr{B}_i$ be an $(n-1)$-graph of girth at least $\ell+1$ with vertex set W_i and as many edges as possible. There are positive constants γ and δ, depending only on n and ℓ, such that $|\mathscr{B}_i|>\gamma|W_i|^{1+\delta}$. This last statement is justified as follows. For $n=3$, the statement is true when $\ell=3$ since the graph is the complete bipartite graph with equal parts, and when $\ell=4$ or 5, the reader is referred to a theorem of Abbott and Zhou [5]. For $n>3$, the statement is a consequence of a theorem of Erdős and Hajnal [7, Theorem 13.3]. [1]

Choose W_i so large that $|\mathscr{B}_i|\geqslant|V_i|$ and choose it to be a smallest set for which this is so. Since $W=\bigcup_{i=1}^k W_i$ is to be a subset of E, we choose t as the least integer such that $t(n-1)+1\geqslant|W|$. Let $w=|W|$.

Observe that

$$w=\sum_{i=1}^{k}|W_i|\leqslant k\max_{1\leqslant i\leqslant k}|W_i|$$

and

$$\left|\bigcup_{i=1}^{k}\mathscr{B}_i\right|=\sum_{i=1}^{k}|\mathscr{B}_i|>\max_{1\leqslant i\leqslant k}|\mathscr{B}_i|\geqslant\gamma\max_{1\leqslant i\leqslant k}|W_i|^{1+\delta}\geqslant\gamma\left(\frac{w}{k}\right)^{1+\delta}.$$

For $i=1,2,\dots,k$, let $\mathscr{B}_i'$ be a subgraph of $\mathscr{B}_i$ with $|V_i|$ edges and pair off the points of V_i with the edges of $\mathscr{B}_i'$. Let $\mathscr{B}_i''$ be the n-graph whose edges are those of the form $\{u\}\cup B$ where B is the edge of $\mathscr{B}_i'$ that has been paired with $u\in V_i$.

Let $\mathscr{F}$ be the n-graph whose edge set consists of

(1) the edges of $\mathscr{L}-E$,

(2) the edges of $\mathscr{H}_s$,

(3) the edges of $\mathscr{B}_i''$, $i=1,2,\dots,k$.

[1] Note that the theorems mentioned have additional restrictions on the chromatic number of the graph. Such restrictions are irrelevant here.

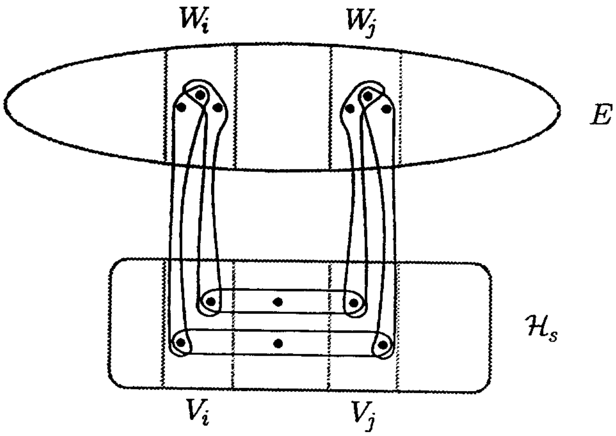

Fig. 3. A cycle of length 6 ($n=3$).

It is not hard to see that $\mathscr{F}$ is $(r+1)$-chromatic and that every edge of $\mathscr{H}_s$ is critical, so that every $(r+1)$-critical subgraph of $\mathscr{F}$ contains $\mathscr{H}_s$.

For $\ell=3,4,5$, $\mathscr{F}$ has girth at least $\ell+1$. There are no cycles of length at most ℓ involving only edges of $\mathscr{L}-E$ or only edges of $\mathscr{H}_s$. Any cycle of length at most ℓ must involve at least two edges of $\bigcup_{i=1}^{k}\mathscr{B}_i''$. The worst that one could have is a cycle of length 6 of the type shown in Fig. 3.

Let $\widehat{\mathscr{F}}$ be an $(r+1)$-critical subgraph of $\mathscr{F}$. Then

$$|V(\widehat{\mathscr{F}})|\geqslant|V(\mathscr{H}_s)|=a^s m+\frac{a^s-1}{a-1}$$

and

$$\begin{aligned}|\widehat{\mathscr{F}}| &\leqslant |\mathscr{L}-E|+|\mathscr{H}_s|+|V(\mathscr{H}_s)|\\ &= t(|\mathscr{G}|-1)+a^sE(m,n,r,\ell)+b\left(\frac{a^s-1}{a-1}\right)+a^sm+\frac{a^s-1}{a-1},\end{aligned}$$

imply

$$\frac{|\widehat{\mathscr{F}}|}{|V(\widehat{\mathscr{F}})|}\leqslant\frac{t(|\mathscr{G}|-1)}{|V(\mathscr{H}_s)|}+\frac{a^sE(m,n,r,\ell)+b((a^s-1)/(a-1))}{a^sm+(a^s-1)/(a-1)}+1.$$

Recall that W_i was chosen as the *smallest* set such that $|\mathscr{B}_i|\geqslant|V_i|$. If $|V_i|=1$, then $\mathscr{B}_i$ has just a single edge, so that $|\mathscr{B}_i|=1=|V_i|$. If $|V_i|\geqslant 2$, then $|\mathscr{B}_i|\geqslant 2$ and hence $|W_i|\geqslant 2n-3$. Let $d(x)$ denote the number of edges of $\mathscr{B}_i$ containing x. Then, when $|V_i|\geqslant 2$,

$$\sum_{x\in W_i} d(x)=(n-1)|\mathscr{B}_i|.$$

Thus, for some $y\in W_i$,

$$d(y)\leqslant\left\lfloor\frac{(n-1)|\mathscr{B}_i|}{|W_i|}\right\rfloor.$$

It follows that the subgraph of $\mathscr{B}_i$ consisting of those edges not containing y has at least $|\mathscr{B}_i| - \lfloor[(n-1)|\mathscr{B}_i|]/|W_i|\rfloor$ edges. We must have $|\mathscr{B}_i| - \lfloor[(n-1)|\mathscr{B}_i|]/|W_i|\rfloor < |V_i|$ for otherwise we have made the wrong choice for W_i. Since $|W_i| \geqslant 2n-3$, and since $(n-1)/(2n-3) \leqslant \frac{2}{3}$ for $n \geqslant 3$, we get $|\mathscr{B}_i| < 3|V_i|$. This last inequality holds when $|V_i| = 1$ as well. Thus, we have

$$t \leqslant \left\lceil \frac{w-1}{n-1} \right\rceil \leqslant \frac{2w}{n} \leqslant \frac{2k}{n}\left(\frac{1}{\gamma}\sum_{i=1}^{k}|\mathscr{B}_i|\right)^{1/(1+\delta)}$$

$$< \frac{2k}{n}\left(\frac{3}{\gamma}\sum_{i=1}^{k}|V_i|\right)^{1/(1+\delta)} = \frac{2k}{n}\left(\frac{3}{\gamma}|V(\mathscr{H}_s)|\right)^{1/(1+\delta)}.$$

It now follows that as $s \to \infty$, $t(|\mathscr{G}|-1)/|V(\mathscr{H}_s)| \to 0$.

Thus,

$$\alpha(n, r+1, \ell) \leqslant \lim_{s\to\infty} \frac{a^s E(m,n,r,\ell) + b((a^s-1)/(a-1))}{a^s m + (a^s-1)/(a-1)} + 1$$

$$= \frac{E(m,n,r,\ell) + b/(a-1)}{m + 1/(a-1)} + 1.$$

Now, let $m \to \infty$. Since a, b are fixed, we get, for $\ell = 3,4,5$, $\alpha(n, r+1, \ell) \leqslant \alpha(n, r, \ell) + 1$. □

Upper bounds for $\alpha(n,r,\ell)$ for various values of n,r,ℓ may be extracted from Theorem 2 and results in the earlier literature. For example, we get the following two corollaries:

Corollary 3. $\alpha(n,r,3) \leqslant r-2$ *for* $r \geqslant 3$, $n \geqslant 3$.

Corollary 4. $\alpha(3,r,\ell) \leqslant r-2$ *for* $r \geqslant 3$, $\ell = 4,5$.

It seems unlikely that one will be able to determine the value of $\alpha(n,r,\ell)$ for all values of n,r,ℓ. Two less ambitious questions, already alluded to, are the following: Can Theorems 1 and 2 be extended to cover $\ell \geqslant 6$? Is it true that $\alpha(n,3,\ell) = 1$ holds for all $n \geqslant 3$, $\ell \geqslant 4$?

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